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Persamaan Diferensial Biasa (C) halaman 167

1. Dapatkan pergerakan sistem pegas-massa yang dihubungkan secara seri di mana diketahui:

$k_1 = 3$ N/m $\qquad k_2 = 4$ N/m

$m_1 = 1$ kg $\qquad m_2 = 4/3$ kg

Jawab :

$F_1 = -k_1 x + k_2 (y-x)$

$F_2 = -k_2 (y-x)$

$\Rightarrow$ •> $m_1 \dfrac{d^2x}{dt^2} = -k_1 x + k_2 (y-x)$

•> $m_2 \dfrac{d^2y}{dt^2} = -k_2 (y-x)$

misal : $x_1 = x$

$x_2 = x' = x_1' \longrightarrow x_2' = x''$

$y_1 = y$

$y_2 = y' = y_1' \longrightarrow y_2' = y''$

$\Rightarrow m_1 \dfrac{d^2x}{dt^2} = -k_1 x + k_2 (y-x)$

$\dfrac{d^2x}{dt^2} = -3x + 4(y-x)$

$\dfrac{d^2x}{dt^2} = -7x + 4y$

$x_2' = -7x_1 + 4y_1$

$\Rightarrow m_2 \dfrac{d^2y}{dt^2} = -k_2 (y-x)$

$\dfrac{\cancel{4}}{3} \dfrac{d^2y}{dt^2} = -\cancel{4}(y-x)$

$y_2' = -3y_1 + 3x_1$

$$\begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -7 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Agar mendapatkan hasil pengelesaian non-trivialnya, maka :

$|\lambda I - A| = 0$

$|\lambda I - A| = 0$

$$\begin{vmatrix} \lambda & -1 & 0 & 0 \\ 7 & \lambda & -4 & 0 \\ 0 & 0 & \lambda & -1 \\ -3 & 0 & 3 & \lambda \end{vmatrix} = 0$$

$$= \lambda \begin{vmatrix} \lambda & -4 & 0 \\ 0 & \lambda & -1 \\ 0 & 3 & \lambda \end{vmatrix} + 1 \begin{vmatrix} 7 & -4 & 0 \\ 0 & \lambda & -1 \\ -3 & 3 & \lambda \end{vmatrix} = 0$$

$$= \lambda\left[\lambda \begin{vmatrix} \lambda & -1 \\ 3 & \lambda \end{vmatrix}\right] + 1\left[7 \begin{vmatrix} \lambda & -1 \\ 3 & \lambda \end{vmatrix} + 4 \begin{vmatrix} 0 & -1 \\ -3 & \lambda \end{vmatrix}\right] = 0$$

$= \lambda^2(\lambda^2+3) + 7(\lambda^2+3) + 4.(-3) = 0$

$= (\lambda^2+7)(\lambda^2+3) - 12 = 0$

$= \lambda^4 + 10\lambda^2 + 21 - 12 = 0$

$= \lambda^4 + 10\lambda^2 + 9 = 0$

$= (\lambda^2+9)(\lambda^2+1) = 0$

$\lambda^2 = -9 \quad \vee \quad \lambda^2 = -1$

$\lambda = \pm 3i \qquad \lambda = \pm i$

▶ Menentukan Vektor Eigen.

•> $\lambda = 3i$

$$\begin{bmatrix} 3i & -1 & 0 & 0 \\ 7 & 3i & -4 & 0 \\ 0 & 0 & 3i & -1 \\ -3 & 0 & 3 & 3i \end{bmatrix} v = 0$$

• $(3i)v_1 - v_2 = 0$

• $7v_1 + (3i)v_2 - 4v_3 = 0$

• $(3i)v_3 - v_4 = 0$

• $-3v_1 + 3v_3 + (3i)v_4 = 0$

$\rightarrow$ ※ $v_2 = (3i)v_1$ $\qquad$ ※ $7v_1 + 3i\,v_2 - 4v_3 = 0$

※ $(3i)v_3 = v_4$ $\qquad$ $7v_1 + 9(-1)v_1 - 4v_3 = 0$

※ $-3v_1 + 3v_3 + (3i)v_4 = 0$ $\qquad$ $-2v_1 = 4v_3$

$6v_3 + 3v_3 + 3i\,v_4 = 0$ $\qquad$ $v_1 = -2v_3$

$9v_3 + 3i\,v_4 = 0$

$i v_4 = -3v_3$

• misal : $v_4 = 3s$ $\qquad$ $v_1 = -2(-is)$

$v_3 = -is$ $\qquad$ $= 2is$

$v_2 = 3i(2i)s$

$= -6s$

Saat $\lambda = 3i$, maka vektor eigennya adalah :

$$v = \begin{bmatrix} 2i \\ -6 \\ -i \\ 3 \end{bmatrix} s$$

•> $\lambda = i$

$$\begin{bmatrix} i & -1 & 0 & 0 \\ 7 & i & -4 & 0 \\ 0 & 0 & i & -1 \\ -3 & 0 & 3 & i \end{bmatrix} v = 0$$

•> $i v_1 - v_2 = 0$

$v_2 = i v_1$

•> $7v_1 + i v_2 - 4v_3 = 0$

$7v_1 - v_1 - 4v_3 = 0$

$6v_1 - 4v_3 = 0$

$3v_1 = 2v_3$

•> $i v_3 - v_4 = 0$

$v_4 = i v_3$

•> $-3v_1 + 3v_3 + i v_4 = 0$

$-2v_3 + 3v_3 + i v_4 = 0$

$v_3 = -i v_4$

Misal : $v_4 = 3s$ $\qquad 2v_3 = 3v_1$

$v_3 = -3is$ $\qquad 2(-\cancel{3}i)s = \cancel{3}v_1$

$v_1 = -2is$

$v_2 = i v_1$

$= i(-2i)s$

$= 2s$

Vektor eigennya :

$$v = \begin{bmatrix} -2i \\ 2 \\ -3i \\ 3 \end{bmatrix} s$$

▶ Saat $\lambda = i$ dan vektor : $\begin{bmatrix} -2i \\ 2 \\ -3i \\ 3 \end{bmatrix}$

$$\bar{x}_1 = e^{(0+i)t} \begin{bmatrix} -2i \\ 2 \\ -3i \\ 3 \end{bmatrix}$$

$$= e^{0t} (\cos t + i \sin t) \begin{bmatrix} -2i \\ 2 \\ -3i \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2i\cos t + 2\sin t \\ 2\cos t + 2i\sin t \\ -3i\cos t + 3\sin t \\ 3\cos t + 3i\sin t \end{bmatrix}$$

$$= \begin{bmatrix} 2\sin t \\ 2\cos t \\ 3\sin t \\ 3\cos t \end{bmatrix} + i \begin{bmatrix} -2\cos t \\ 2\sin t \\ -3\cos t \\ 3\sin t \end{bmatrix}$$

▶ Saat $\lambda = 3i$ dan vektor eigen = $\begin{bmatrix} 2i \\ -6 \\ -i \\ 3 \end{bmatrix}$

$$\bar{x}_2 = e^{(0+3i)t} \begin{bmatrix} 2i \\ -6 \\ -i \\ 3 \end{bmatrix}$$

$$= e^{0t} (\cos 3t + i \sin 3t) \begin{bmatrix} 2i \\ -6 \\ -i \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2i\cos 3t - 2\sin 3t \\ -6\cos 3t - 6i\sin 3t \\ -i\cos 3t + \sin 3t \\ 3\cos 3t + 3i\sin 3t \end{bmatrix}$$

$$= \begin{bmatrix} -2\sin 3t \\ -6\cos 3t \\ \sin 3t \\ 3\cos 3t \end{bmatrix} + i \begin{bmatrix} 2\cos 3t \\ -6\sin 3t \\ -\cos 3t \\ 3\sin 3t \end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\\y_1\\y_2\end{bmatrix} = C_1\begin{bmatrix}2\sin t\\2\cos t\\3\sin t\\3\cos t\end{bmatrix} + C_2\begin{bmatrix}-2\cos t\\2\sin t\\-3\cos t\\3\sin t\end{bmatrix} + C_3\begin{bmatrix}-2\sin 3t\\-6\cos 3t\\\sin 3t\\3\cos 3t\end{bmatrix} + C_4\begin{bmatrix}2\cos 3t\\-6\sin 3t\\-\cos 3t\\3\sin 3t\end{bmatrix}$$

$X_1 = x = 2C_1\sin t + (-2)C_2\cos t - 2C_3\sin 3t + 2C_4\cos 3t$

$y_1 = y = 3C_1\sin t - 3C_2\cos t + C_3\sin 3t - C_4\cos 3t$

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2. Diketahui: $V_1 = 60\, l$ $\quad A_1(0) = 60$ gr

$V_2 = 60\, l$ $\quad A_2(0) = 200$ gr

$r_{in} = 3\text{ gr}/l$ $\quad C_{in} = 6\ l/\text{mnt}$ $\quad r_{21} = 8$

$r_{12} = 2$ $\quad r_{out} = 6$

Ditanya = Selesaikan masalah pencampuran

Jawab =

$\Rightarrow \dfrac{dA_1}{dt} = -\dfrac{r_{21}}{V_1}A_1 + \dfrac{r_{12}}{V_2}A_2 + C_{in}r_{in}$

$= \dfrac{-8}{60}A_1 + \dfrac{2}{60}A_2 + 6.3$

$= -\dfrac{2}{15}A_1 + \dfrac{1}{30}A_2 + 18$

$\Rightarrow \dfrac{dA_2}{dt} = \dfrac{r_{21}}{V_1}A_1 - \dfrac{r_{21}}{V_2}A_2$

$= \dfrac{8}{60}A_1 - \dfrac{8}{60}A_2$

$= \dfrac{2}{15}A_1 - \dfrac{2}{15}A_2$

$\Rightarrow$ Kondisi awal: $\begin{pmatrix}A_1(0)\\A_2(0)\end{pmatrix} = \begin{pmatrix}60\\200\end{pmatrix}$

▶ Sistem PD : $x' = A\bar{x} + b$

dengan : $A = \begin{pmatrix}-2/15 & 1/30\\ 2/15 & -2/15\end{pmatrix}$, $b = \begin{pmatrix}18\\6\end{pmatrix}$

$\text{Det}(\lambda I - A) = 0$

$$\begin{vmatrix}\lambda + \frac{2}{15} & -\frac{1}{30}\\ -\frac{2}{15} & \lambda + \frac{2}{15}\end{vmatrix} = 0$$

$\left(\lambda + \frac{2}{15}\right)^2 - \frac{2}{450} = 0$

$\lambda^2 + \frac{4}{15}\lambda + \frac{4}{225} - \frac{2}{450} = 0$

$\lambda^2 + \frac{4}{15}\lambda + \frac{6}{450} = 0$

$450\lambda^2 + 120\lambda + 6 = 0$

$150\lambda^2 + 40\lambda + 2 = 0$

$75\lambda^2 + 20\lambda + 1 = 0$

$(15\lambda + 1)(5\lambda + 1) = 0$

$\lambda = -\frac{1}{15} \vee \lambda = -\frac{1}{5}$

▶ Saat $\lambda = -\frac{1}{15}$

$= \begin{pmatrix}1/15 & -1/30\\ -2/15 & 1/15\end{pmatrix} v = 0$

$\xrightarrow{2b_1 + b_2} \begin{pmatrix}1/15 & -1/30\\ 0 & 0\end{pmatrix} v = 0$

$\frac{1}{15}v_1 - \frac{1}{30}v_2 = 0$ $\quad$ misal : $v_1 = s$

$2v_1 - v_2 = 0$ $\quad$ $v_2 = 2s$

$v_2 = 2v_1$

$V_1 = \begin{pmatrix}1\\2\end{pmatrix}s$

▶ Saat $\lambda = -\frac{1}{5}$

$= \begin{pmatrix}-1/15 & -1/30\\ -2/15 & -1/15\end{pmatrix} v = 0$

$\xrightarrow{-2b_1 + b_2} \begin{pmatrix}-1/15 & -1/30\\ 0 & 0\end{pmatrix} v = 0$

$-\frac{1}{15}v_1 - \frac{1}{30}v_2 = 0$ $\quad$ misal : $v_1 = s$

$2v_1 + v_2 = 0$ $\quad$ $v_2 = -2s$

$v_2 = -2v_1$

$V_2 = \begin{pmatrix}1\\-2\end{pmatrix}s$

$X_c : C_1 e^{-1/15\,t}\begin{pmatrix}1\\2\end{pmatrix} + C_2 e^{-1/5\,t}\begin{pmatrix}1\\-2\end{pmatrix}$

matriks fundamental : $X = \begin{pmatrix}e^{-1/15\,t} & e^{-1/5\,t}\\ 2e^{-1/15\,t} & -2e^{-1/5\,t}\end{pmatrix}$

$X_p = XU$

$XU' = b$

$$xu' = b$$

$$\begin{pmatrix} e^{-\frac{1}{15}t} & e^{-\frac{1}{5}t} \\ 2e^{-\frac{1}{15}t} & -2e^{-\frac{1}{5}t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 18 \\ 0 \end{pmatrix}$$

$$\begin{matrix} \underbrace{e^{1/15 t}(b_1)} \\ \frac{1}{2}e^{1/15t}(b_2 \end{matrix} \left(\begin{array}{cc|c} 1 & e^{-2/15 t} & 18e^{1/15 t} \\ 1 & -e^{-2/15 t} & 0 \end{array}\right)$$

$$\underbrace{-b_1+b_2} \left(\begin{array}{cc|c} 1 & e^{-2/15t} & 18e^{1/15t} \\ 0 & -2e^{-2/15t} & -18e^{1/15t} \end{array}\right)$$

$$-2e^{-2/15t} u_2' = -18e^{1/15t}$$

$$u_2' = 9e^{1/5t}$$

$$u_1' + e^{-2/15t} u_2' = 18e^{1/15t}$$

$$u_1' + e^{-2/15t} \cdot 9e^{1/5t} = 18e^{1/15t}$$

$$u_1' + 9e^{1/15t} = 18e^{1/15t}$$

$$u_1' = 9e^{1/15t}$$

$u_1' = 9e^{1/15t}$	$u_2' = 9e^{1/5t}$
$u_1 = \int 9e^{1/15t}\,dt$	$u_2 = \int 9e^{1/5t}$
$= 135e^{1/15t}$	$= 45e^{1/5t}$

$$X_p = Xu$$

$$= \begin{bmatrix} e^{-1/15t} & e^{-1/5t} \\ 2e^{-1/15t} & -2e^{-1/5t} \end{bmatrix} \begin{bmatrix} 135e^{1/15t} \\ 45e^{1/5t} \end{bmatrix}$$

$$= \begin{bmatrix} 135+45 \\ 270-90 \end{bmatrix}$$

$$= \begin{bmatrix} 180 \\ 180 \end{bmatrix}$$

$$\text{PUPD} = y_c + y_p$$

$$= C_1e^{-1/15t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2e^{-1/5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{bmatrix} 180 \\ 180 \end{bmatrix}$$

▶ Menentukan PKPD dengan substitusi kondisi awal

$$X(0) = \begin{pmatrix} 60 \\ 200 \end{pmatrix}$$

$$X(0) = C_1e^0\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{bmatrix} 180 \\ 180 \end{bmatrix} = \begin{bmatrix} 60 \\ 200 \end{bmatrix}$$

$C_1 + C_2 = -120$	$C_1 + C_2 = -120$
$2C_1 - 2C_2 = 20$	$C_1 - C_2 = 20$
	$2C_2 = -140$
	$C_2 = -70$

$$\text{※ } C_1 = -120 + 70$$

$$= -50$$

PKPD :

$$X(t) = -50e^{-1/15t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 70e^{-1/5t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{bmatrix} 180 \\ 180 \end{bmatrix}$$

$$A_1 = -50e^{-1/15t} - 70e^{-1/5t} + 180$$

$$= 180 - 50e^{-1/15t} - 70e^{-1/5t}$$

$$A_2 = -100e^{-1/15t} + 140e^{-1/5t} + 180$$

$$= 180 + 140e^{-1/5t} - 100e^{-1/15t}$$

Jawab :

$$A_1 = 180 - 70e^{-t/5} - 50^{-t/15}$$

$$A_2 = 180 + 140e^{-t/5} - 100e^{-t/15}$$

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